A QUASI-ANALYTICAL METHOD FOR PERIOD-DOUBLING BIFURCATION

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ABSTRACT
Prediction of period-doubling bifurcation is accomplished very accurately by using higher-order Harmonic Balance Approximations (HBAs) and quasi-analytical monodromy-matrix evaluation. Approximation error analysis is carried out for the computation. An accurate detection of first period-doubling bifurcation in Chua’s circuit is demonstrated.

1. INTRODUCTION
The aim of this paper is to show how period-doubling bifurcation, emerging close to a Hopf bifurcation, can be predicted with great accuracy. The prediction is accomplished using a relatively small number of harmonics in the HBA. It is well known that the cascade route of period-doubling bifurcations is a very important scenario to understand chaotic motions. In the realm of nonlinear circuits and systems, there are some notable recent contributions [2, 4, 6, 7, 9]; some of them [2, 7, 9] proposed to use the describing function method (generally a first-order harmonic analysis), while some others [4] suggested to use the harmonic balance approach with several harmonics.

In this paper, we propose a more accurate computational scheme for predicting period-doubling bifurcation, by taking advantage of some explicit formulas of periodic solutions in terms of higher-order HBAs [8], and a time domain polynomial expression. This new scheme improves the accuracy of the detection as compared to other closely related approaches [2, 7]. It also reveals some insights about the hidden relationships of period-doubling mechanisms and the coefficients of the Fourier series in quasi-analytical computations. The proposed algorithm uses the information of higher-order HBAs and the evaluations (based in the shadowing lemma) of the associated monodromy matrix and an error index. This algorithm provides approximate characteristic multipliers (Floquet multipliers), without using too many harmonics in comparison with the method recently proposed in [4]. Thus, our approach provides significant savings of computational efforts. In addition, since the new algorithm can be implemented in a symbolic way, it can be used in the future to study the structure of a type of normal form for period-doubling bifurcations that are close to Hopf bifurcations. Moreover, a detailed analysis of the approximation errors in the new approach has been carried out completing previous results [9].

2. THE GRAPHICAL METHOD FOR HOPF BIFURCATION ANALYSIS
Consider the following Lur’e system:
\[ \dot{x}(t) = f(x; \mu) = A(\mu)x(t) + B(\mu)g(C(\mu)x(t)), \]
\[ y(t) = C(\mu)x(t), \]
where \(A, B\) and \(C\) are \(n \times n, n \times r\) and \(m \times n\) matrices, respectively, \(\mu \in \mathbb{R}\) is the bifurcation control parameter, \(x \in \mathbb{R}^n\) is the state vector, \(y \in \mathbb{R}^m\) is the system output, \(g : \mathbb{R}^n \to \mathbb{C}^{2r+1}(\mathbb{R}^r)\) is the system feedback (a smooth nonlinear function), \(f : \mathbb{R}^n \to \mathbb{C}^{2r+1}(\mathbb{R}^r)\) is a smooth system vector field, and \(n, m, q, r\) are positive integers. Define
\[ G(\xi; \mu) = C(\mu) [i A(\mu)]^{-1} B(\mu), \]
and let \(z = -\dot{z}\) with \(g(x; \mu) := h(z; \mu)\). Then, the equilibrium solution of (1) can be obtained by solving the following equation:
\[ G(0; \mu) h(z; \mu) = -\dot{z}. \]
Also, define the Jacobian \(J_i = \frac{\partial h(z)}{\partial z_i} \mid_{z=0}\).

When a Hopf bifurcation occurs, one of the eigenvalues of the corresponding linearized system transfer matrix, \(G(\xi; \mu) J_i\), denoted \(\lambda\), satisfies
\[ \lambda(i \omega_0; \mu_0) = -1 + 0i, \quad i = \sqrt{-1}, \]
for some values of \(i \omega_0\) and \(\mu_0\). At the moment of bifurcation, a periodic branch arises from criticality, and continues to develop as \(\mu\) is varied. Then, a \(q\)-th-order approximate periodic solution of Eq. (1) can be written as
\[ z(t) \approx z_0(t) = \dot{z} + \mathbb{R}\left\{ \sum_{k=0}^{q} \sum_{\nu=0}^{q} Z_k^{\nu} e^{i(k \nu \omega_0 t)} \right\}, \quad q = 1, ..., 4, \]
to distinguish from the true solution \(x_H(t) = z(t) = \sum_{k=0}^{q} \mathbb{R}\left\{ \sum_{\nu=0}^{q} Z_k^{\nu} e^{i(k \nu \omega_0 t)} \right\}\), where \(\mathbb{R}\) is the real part, \(\omega_H\) is the fundamental frequency, \(\omega_0\) is the approximation frequency, and \(Z_k^{\nu}\) are the \(k\)-harmonic complex amplitudes satisfying the harmonic balance equations
\[ Z_k^{\nu} = -G(i k \omega_0; \mu) H_k^{\nu}, \quad k \in \{0, 1, 2, \ldots\}, \]
where \(\{H_k^{\nu}\}\) are the Fourier coefficients of the output signal of the nonlinear feedback \(h(z(t)) = \mathbb{R}\left\{ \sum_{k=0}^{\infty} H_k^{\nu} e^{i(k \nu \omega_0 t)} \right\}\).
written as polynomial functions of \( \{Z^k\} \). The Graphical Hopf Bifurcation Method (GHBM) [8] provides the \( \ell \)-th order prediction of the limit cycle. Here, only the first \( 2q+1 \) Fourier coefficients \( \{H^k_q\} \) of the output signal of the nonlinear feedback, written as polynomial functions of \( \{Z^k\} \), are considered:

\[
h(z(t)) = \Re \left\{ \sum_{k=0}^{2q} H^k_q e^{i k \omega_0 t} \right\} + \Re \left\{ \sum_{k=2q+1}^{\infty} H^k_q e^{i k \omega_0 t} \right\}.
\]

These equations are solved in terms of \( Z^k = Z^k_0(v, \theta_0) \), where \( v \) is the right eigenvector of \( G(\omega, \mu)J_1 \) associated with the eigenvalue \( \lambda \), and \( \theta_0 \) is a measure of the amplitude of the periodic solution. More details about this method can be found in [3,8].

3. STABILITY ANALYSIS OF THE LIMIT CYCLES

Suppose a limit cycle, \( x_H(t) \), has been generated by the Hopf bifurcation mechanism. Define a perturbed orbit, \( x_p(t) \), by

\[
x_p(t) = x_H(t) + x_D(t),
\]

where \( x_D(t) \) is a perturbation of \( x_H(t) \). Taking a time derivative gives

\[
\dot{x}_p(t) = \dot{x}_H(t) + \dot{x}_D(t) = A(x_H + x_D) + Bg[C(x_H + x_D)].
\]

Then, it follows that

\[
\dot{x}_D(t) = Ax_D(t) + B\{g[C(x_H + x_D)] - g(Cx_H)\}.
\]

It is then possible to demonstrate the growth of decay of \( x_D(t) \) and, therefore, the stability or instability of \( x_H(t) \) by using an appropriate Poincaré section and looking for a fixed point solution of the so-called return map. The stability of this fixed point depends on the positions of the eigenvalues (or characteristic multipliers) of an associate matrix, \( M \), called the monodromy matrix, with respect to the unit circle. The monodromy matrix \( M \) is defined by the following matrix differential system:

\[
\begin{align*}
\dot{X}(t) &= J_D(t)X(t), \\
X(0) &= I, \\
M &= X \left( \frac{2\pi}{\omega_0} \right),
\end{align*}
\]

where \( J_D(t; \mu) \) is a periodic matrix defined by

\[
J_D(t) = \left. \frac{\partial \dot{x}_D(t)}{\partial x_D} \right|_{x_D=0} = A + B \left. \frac{\partial g(x)}{\partial x} \right|_{x=x_H(t)} = A + BJ_{x_H}(t).
\]

The characteristic multipliers are denoted as \( \lambda_i, \ i = 1, \ldots, n \), where one of them, \( \lambda_1 \), is always equal to \( +1 \). As to the stability, a periodic orbit becomes stable when all the characteristic multipliers (except the one at \( +1 \)) stay within the unit circle, and it becomes unstable when they move out. Here, we are interested when \( \lambda_1 \) crosses the border \(-1 + i0\), i.e. the period-doubling bifurcation condition.

One problem in the application of matrix \( M \) is the need to integrate two dynamical systems simultaneously: the original nonlinear system and the variational equation (6). In order to drastically reduce the computational burden, we have proposed in [3] to calculate the following approximate matrices \( M_q \) (when \( 2q \) are the numbers of harmonics of the periodic solution), instead of the original \( M \), so that we only need to deal with the integration of the (approximate) variational equation:

\[
\begin{align*}
\dot{Y}(t) &= J_{D_q}(t)Y(t), \\
Y(0) &= I, \\
M_q &= Y \left( \frac{2\pi}{\omega_0} \right),
\end{align*}
\]

where \( J_{D_q}(t) \) is a periodic matrix \( J_{D_q}(t) = A + BJ_{x_H}(t) \), with \( J_{x_H}(t) = \left. \frac{\partial g(x)}{\partial x} \right|_{x=x_H(t)} \) obtained by using the information of \( 2q \) harmonics of HBAs.

4. ANALYSIS ON APPROXIMATION ERRORS

In this section, we provide some formulas to deal with the approximation error in a quasi-analytical manner. This distinguishes our approach from many others in the current literature, which generally do not carry out evaluation of computational or approximation errors.

4.1. A Taylor series expansion

Consider the following parametrized nonlinear system:

\[
x(t) = f(x(t), t; \mu), \quad x(t_0) = x_0.
\]

For notational convenience below, we denote \( g_i(u, t; \mu) := f(u, t; \mu) \). Note that

\[
dx(t) dt \bigg|_{t=t_0} = g_i(x(t), t; \mu)|_{t=t_0} = f(x(t_0), t_0; \mu).
\]

and in general, it is easy to deduce that

\[
d^k x(t) dt^k \bigg|_{t=t_0} = g_i(x(t), t)|_{t=t_0} \frac{\partial g_i}{\partial x} \bigg|_{x=x_0} dx(t) dt
\]

Thus, the Taylor-series expansion of a smooth function \( x(t) \), if it converges, takes the form

\[
x(t) = \sum_{k=0}^{\infty} a_k (t-t_0)^k, \quad x(t_0) = x_0
\]

where the coefficients \( a_k \) satisfy \( x(t_0) = a_0 \), and \( \frac{dx(t)}{dt} \bigg|_{t=t_0} = a_1 \). To this end, it is easy to derive the following expressions:

\[
a_0 = x_0,
\]

\[
a_k = \frac{\partial a_{k-1}}{\partial x_0} f(x_0, t_0) + \frac{\partial a_{k-1}}{\partial t_0}, \quad k > 0,
\]

and the corresponding Taylor series can be written as

\[
x(t) = x_0 + \left[ \sum_{k=1}^{\infty} \frac{\partial a_{k-1}}{\partial x_0} (t-t_0)^k \right] f(x_0, t_0)
\]

\[
+ \left[ \sum_{k=1}^{\infty} \frac{\partial a_{k-1}}{\partial t_0} (t-t_0)^k \right].
\]
Note that $a_k$ depends only on $a_0, \ldots, a_{k-1}$, so evaluations of formulas (8) and (9) are straightforward, which can be completed either analytically or numerically.

### 4.2. Evaluation of the approximation error

Suppose we approximate the solution of Eq. (1) by its $r$th-order truncation

$$x_r(t) = \sum_{k=0}^{r} a_k \frac{(t-t_0)^k}{k!}.$$  

Here, we note that the polynomial has the same coefficients $a_0, a_1, \ldots, a_r$ of the original ODE solution, and that $\dot{x}_r \approx f(x_r(t), t; \mu)$ which becomes equal if and only if $r = \infty$. Thus, it is reasonable to introduce an error measure (or error index) defined by

$$\Delta_r = \|f(x_r(t), t; \mu) - \dot{x}_r(t)\|.$$

Obviously, $\Delta_r \to 0$ as $r \to \infty$. In order to get more insights about this measure, rewrite

$$\Delta_r = \|\dot{x}(t) - \dot{x}_r(t) + f(x_r(t), t; \mu) - f(x(t), t; \mu)\|$$

and then apply the triangular inequality to it, giving rise to

$$\Delta_r \leq \|\dot{x}(t) - \dot{x}_r(t)\| + \|f(x_r(t), t; \mu) - f(x(t), t; \mu)\|.$$

Then, by applying the Mean Value Theorem, we obtain

$$\|f(x_r(t), t; \mu) - f(x(t), t; \mu)\| = \left| \frac{\partial f(u, t; \mu)}{\partial u} \right|_{u=u_0} \|x_r(t) - x(t)\|,$$

where, for each $t$, $u_0 = \alpha x_r(t) - (1 - \alpha) x(t)$ for some $\alpha \in [0, 1]$. Thus,

$$\Delta_r \leq \|\dot{x}(t) - \dot{x}_r(t)\| + \left| \frac{\partial f(u, t; \mu)}{\partial u} \right|_{u=u_0} \|x_r(t) - x(t)\|,$$

which can be further elaborated to yield

$$\Delta_r \leq \hat{\Delta}_r := \sum_{k=r+1}^{\infty} a_k \frac{(t-t_0)^k}{(k-1)!} + \left| \frac{\partial f(u, t; \mu)}{\partial u} \right|_{u=u_0} \sum_{k=r+1}^{\infty} a_k \frac{(t-t_0)^k}{k!}.$$

Now, notice that $\hat{\Delta}_r$ reduces to zero if either one of the following conditions is satisfied:

- $t = t_0$,
- $a_k = 0$, $k > r$.

Also, note that because the original Taylor series converges, we have $\lim_{r \to \infty} a_r = 0$, so that $\lim_{r \to \infty} \Delta_r = 0$.

The above formulation prepares the following evaluation and analysis on polynomial approximation errors.

### Remarks

Define

$$\begin{align*}
a_{0,0} &= x_0, \\
a_{0,k} &= \frac{\partial a_{0,k-1}}{\partial a_{0,0}} f(a_{0,0}, t_0) + \frac{\partial a_{0,k-1}}{\partial t_0}.
\end{align*}$$

Suppose that we use the approximation

$$x_{r,0}(t_1) = \hat{x}_1(t_1) = \sum_{k=0}^{r} a_{0,k} \frac{(t_1 - t_0)^k}{k!},$$

in order to get a point, $\hat{x}_1(t_1)$, close enough to the “true” point

$$x(t_1) = \sum_{k=0}^{\infty} a_{1,k} \frac{(t_1 - t_0)^k}{k!}.$$  

Then, the question is: Can we actually obtain an approximation,

$$\begin{align*}
x_{r,1}(t) &= \sum_{k=0}^{r} a_{1,k} \frac{(t - t_1)^k}{k!}, \\
a_{1,0} &= \hat{x}_1(t_1), \\
a_{1,k} &= \frac{\partial a_{0,k-1}}{\partial a_{1,0}} f(a_{1,0}, t_1) + \frac{\partial a_{1,k-1}}{\partial t_1},
\end{align*}$$

such that it is close enough to the true solution $x(t)$? Fortunately, the answer is yes. The Shadowing Lemma implies that while we may not be able to calculate the real solution, we can find a close enough approximation to it. The condition for applying this lemma is that the points $x_0, \hat{x}_1(t_1), \ldots$, belong to an hyperbolic invariant set. This means that the trajectories initiated inside a given neighborhood will never leave it, and that the discrete map inside this neighborhood contracts at an uniformly exponential rate, $\lambda$. So, we are entitled to define a special discrete map and expect to iteratively calculate a sequence of points that are close enough to the true solution of the concerned equation. Here, the key is that the discrete map defined by

$$\begin{align*}
t_m &= t_0 + mt_0, \\
a_0(x_m, t_m) &= x_m, \\
a_k(x_m, t_m) &= \frac{\partial a_{k-1}}{\partial x_m} f(x_m, t_m; \mu) + \frac{\partial a_{k-1}}{\partial t_m}, \\
x_{m+1} &= \sum_{k=0}^{Q} a_k(x_m, t_m) \frac{t_m^k}{k!},
\end{align*}$$

can be evaluated both analytically and numerically, as a function of $x_0$, $t_0$ and $t_Q$, where $x_0$ and $t_0$ are the initial conditions and $t_Q$ is a time step integration constant.

Usually, performing a numerical evaluation is much faster than carrying out an analytical approximation. Then, why should one bother to consider analytical or quasi-analytical calculations? Our answer is that an analytic solution, although often approximate, is a function that has a useful and precise structure. Knowing this structure is often important and it can be applied for controller design, analysis of the normal form for the bifurcation, and so on. Another important point in note is that for small values of $Q$, analytical expressions for the coefficients $a_k$ are quite compact.

An important point is that we can analyze the error from two different points of view: the traditional approach is the evaluation of a quantitative bound on the difference.
between a true solution and a prediction, and a different approach is to analyze (12) and detect which values of $t_0$ generates a qualitative different dynamic respect to (1).

5. AN EXAMPLE

Consider the Chua’s circuit model with a smooth nonlinearity [1]

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} = \begin{bmatrix}
\mu (y - \varphi(x)) \\
x - y + z \\
\alpha (\varphi(x) - \beta y)
\end{bmatrix},
\]

where $\varphi(x) = -\mu (c_3 x + c_4 x^2 + c_1 x^3)$, and $\mu$, $\beta$, $c_1$, $c_2$, $c_3$ and $c_4$ are system parameters.

After choosing a convenient realization, we end up with

\[
G(s, \mu) = \frac{s^2 + s + \beta}{s^3 + s^2(1 + \frac{1}{2}c_3 \mu) + s(\mu + \frac{1}{2}c_3 \mu + \beta) + \frac{1}{2}c_3 \beta \mu}.
\]

\[
J = -\mu \left( -\frac{c_3}{2} + 2c_2 z_1 - 3c_1 z_1^2 \right).
\]

Hopf bifurcation points satisfy the following expressions:

\[
\dot{z}_1 = \pm \sqrt{-c_3 c_1}, \quad \dot{\omega}^2 = -2c_3 (1 - 2c_2) \mu^2.
\]

A period-doubling bifurcation, obtained by numerical integration, gives the critical value $\mu_{PD} = 8.198$ while fixing $c_1 = \frac{1}{10}$, $c_3 = \frac{1}{2}$, $c_2 = c_4 = 0$ and $\beta = 14$. Using the new algorithm, it is possible to compute the characteristic multipliers, as shown in Tables 1, 2 and 3 for the approximate monodromy matrices $M_1$, $M_2$ and $M_3$. The largest difference $\lambda_1 - 1$ is obtained for the $L_1$-approximation. Furthermore, it is easily observed that the increase of the difference $(\lambda_1 - 1)$ is related with the increase in the number of harmonics in the approximation of the limit cycle. Using this quasi-analytical technique, the value of the period-doubling bifurcation is obtained as $\mu_{PD, QP} \approx 8.22$, which is very close to $\mu_{PD} = 8.198$. Using asymptotic averaging techniques, the authors in [10] showed that the $\mu_{PD, QP}$ is $7.81$ which has a larger error compared to our prediction.

6. CONCLUSIONS

A quasi-analytical approach has been developed in this paper for detecting period-doubling bifurcation in a nonlinear dynamical system. Prediction of the period-doubling bifurcation is accomplished very accurately via the proposed computational scheme, with a reasonably small number of harmonics used in the computation. The technique has the advantage of utilizing the structure of the system for analysis, design or other purposes, since the harmonic content as well as the approximate monodromy matrices can be implemented explicitly. This, and several other special features, distinguishes the proposed approach from other existing ones in the literature.

<table>
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<tr>
<th>$\mu$</th>
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<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
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<td>-0.028 - 0.018i</td>
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Table 3: Characteristic multipliers of $M_3$ by varying parameter $\mu$ (others are identical to Table 1).

7. REFERENCES